

A Comment on Jones Inclusions with infinite Index

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Abstract

Given an irreducible inclusion of infinite von-Neumann-algebras $\mathcal{N} \subset \mathcal{M}$ together with a conditional expectation $E : \mathcal{M} \rightarrow \mathcal{N}$ such that the inclusion has depth 2, we show quite explicitly how \mathcal{N} can be viewed as the fixed point algebra of \mathcal{M} w.r.t. an outer action of a compact Kac-algebra acting on \mathcal{M} . This gives an alternative proof, under this special setting, of a more general result of M. Enock and R. Nest, [E-N], see also S. Yamagami, [Ya2].

1 Introduction

In the algebraic approach to quantum field theory as proposed by R. Haag et al. , [Haa], one has the natural inclusion of the observable algebra lying inside the field algebra. Due to the beautiful results of S. Doplicher and J. Roberts, [Do-Ro], at least in four spacetime dimension this inclusion can be described by the outer action of a compact group on the field algebra such that the observables are the fixed point elements under this action. In lower dimension the question how in general the observable algebra sits inside the field algebra is still open (symmetry problem in quantum field theory).

The basic inclusion one starts with is of the type

$$\rho(\mathcal{A}) \subset \mathcal{A},$$

where ρ is a localized endomorphism and \mathcal{A} the quasi local C^* -algebra of observables. Here ρ represents a charged particle of the theory. In this framework antiparticles are also described by localized endomorphisms, the so called conjugate endomorphisms $\bar{\rho}$. They are characterized by the existence of an observable intertwiner $R_{\bar{\rho},\rho} \in \mathcal{A}$, such that

$$R_{\bar{\rho},\rho} \in (id, \bar{\rho} \circ \rho).$$

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Due to the result of R. Longo, [Lo2], the existence of such an intertwiner is equivalent to the existence of a conditional expectation $E : \mathcal{A} \rightarrow \rho(\mathcal{A})$, see also section 2, where we review some of his results.

In order to get a better understanding of how things may look like we investigate a simplified situation, i.e. we consider an irreducible inclusion of infinite von-Neumann-factors $\mathcal{N} \subset \mathcal{M}$ of depth 2, see the definition below, together with a faithful conditional expectation $E : \mathcal{M} \rightarrow \mathcal{N}$. It has first been claimed by A. Ocneanu, [Oc], and is meanwhile well known, [Lo1,Sz,Da], that at least in the finite index case the inclusion stems from the outer action of a Hopf-algebra on \mathcal{M} . In the quantum field theoretic setting of S. Doplicher and J. Roberts, this would imply that the concept of group symmetry gets replaced by a “quantum group symmetry”, where \mathcal{M} would be the field algebra and \mathcal{N} the observable algebra.

It turns out that Longo’s intertwiner is extremely useful in order to identify the Hopf algebra structure together with its dual, the convolution product, Fouriertransform and Haar state etc. and the formulae of A. Ocneanu, [Oc1,2], are easily rephrased in terms which also work in the infinite index case.

The plan of our paper is as follows. Given $E : \mathcal{M} \rightarrow \mathcal{N}$ and Longo’s canonical endomorphism $\gamma : \mathcal{M} \rightarrow \mathcal{N}$ we review in section 2 some basic results of R. Longo, [Lo2,3], on the Jones triple $\gamma(\mathcal{M}) \subset \mathcal{N} \subset \mathcal{M}$ and the associated intertwiner space (id, γ) . We also introduce the relative commutants $\mathcal{A}_{i,j} = \mathcal{M}'_{i-1} \cap \mathcal{M}_j$, where $\mathcal{M}_{2n} = \gamma^{-n}(\mathcal{M})$ and $\mathcal{M}_{2n-1} = \gamma^{-n}(\mathcal{N})$, and we discuss Ocneanu’s depth 2 condition, [Oc1,2], in terms of a Pimsner-Popa basis $\lambda_i \in \mathcal{A}_{-1,0}$. In section 3 we use these data to define a Fourier transform $\mathcal{F} : \mathcal{A}_{-1,0} \rightarrow \mathcal{A}_{-2,-1}$ and a convolution product $*$ on $\mathcal{A}_{-1,0}$. In section 4 we construct the coproduct on $\mathcal{A}_{-1,0}$, the Haar weights on $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ and prove that \mathcal{F} extends to a unitary on the associated L^2 -spaces. We also have natural candidates for involutive antipodes on $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$, respectively, anticipating that $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ become a pair of dual Kac algebras. In section 5 we construct a natural Hopf-module right action of $\mathcal{A}_{0,1} \equiv \gamma^{-1}(\mathcal{A}_{-2,-1})$ on \mathcal{M}_0 such that $\mathcal{M}_1 = \mathcal{M}_0 \triangleleft \mathcal{A}_{0,1}$ becomes a crossed product. In section 6 we construct a multiplicative unitary and use the results of S. Baaß and G. Skandalis, [B-S], to complete the proof of our setting.

As a note on our terminology we remark that we talk of Kac algebras, since the antipodes are involutive. We also call $\mathcal{A}_{0,1}$ a compact Kac algebra, since it contains the Jones projection e_1 as a two-sided integral and therefore generalizes the role of a compact symmetry group (or rather its convolution algebra). This agrees with [Ya3], but opposes [Po-Wo, E-S], where $\mathcal{A}_{0,1}$ would be called “discrete”.

During our investigation we got a preprint of M. Enock and R. Nest, [E-N], where a much more general analysis is given. They look at arbitrary irreducible depth 2 inclusions of factors fulfilling a certain regularity condition and also construct a multiplicative unitary.

In view of their work we do not claim to get any exiting new result. Nevertheless, since their formulas are often quite involved and implicit, it might be helpful to have an alternative approach to our special case. Another advantage of our methods is that many of the formulae we derive work quite well also for reducible inclusions of depth 2. But this is still under investigation, [N-W]. We hope to come back to this in the near future.

2 Preliminaries

We start with introducing the notations and, for the readers convenience, with reviewing some of the results obtained by R. Longo, [Lo2,3], which we will use in the following.

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of von-Neumann factors with separable predual, acting on a Hilbert space $\tilde{\mathcal{H}}$. Let Ω be a common cyclic and separating vector. Denote $J_{\mathcal{M}}, J_{\mathcal{N}}$ the associated modular conjugations and

$$\gamma := \text{Ad } J_{\mathcal{N}} J_{\mathcal{M}}$$

the canonical endomorphism of Longo, see [Lo4]. We consider the Jones tower

$$\mathcal{N} = \mathcal{M}_{-1} \subset \mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \dots$$

where

$$\mathcal{M}_{2n-1} := \gamma^{-n}(\mathcal{N}) \quad , \quad \mathcal{M}_{2n} = \gamma^{-n}(\mathcal{M}),$$

see [Lo2,3]. We will assume the existence of a faithful conditional expectation

$$E_0 : \mathcal{M} \rightarrow \mathcal{N}.$$

By the work of R. Longo, [Lo2], this assumption is equivalent to the existence of an intertwiner $w_{-1} \in (id, \gamma)$, $w_{-1} \in \mathcal{N}$: Take the state

$$m \mapsto \langle \Omega, E_1(m) \Omega \rangle, \quad m \in \mathcal{M}$$

on \mathcal{M} . By modular theory, [St-Zs], we can represent this state uniquely by a vector $\xi \in P^{\natural}(\mathcal{M}, \Omega)$, where $P^{\natural}(\mathcal{M}, \Omega)$ is the natural positivity cone of (\mathcal{M}, Ω) . Then the orthogonal projection

$$e_1 : \mathcal{H} \rightarrow \overline{\mathcal{N}\xi}$$

is called the Jones projection associated with the conditional expectation E_0 . From the definition we get

$$[e_1, J_{\mathcal{M}}] = 0.$$

Let

$$\begin{aligned} w'_{-1} : \mathcal{N}\Omega &\rightarrow \mathcal{N}\xi \\ n\Omega &\mapsto n\xi \quad n \in \mathcal{N}. \end{aligned} \tag{1}$$

Then w'_{-1} obviously defines an isometry in \mathcal{N}' obeying

$$w'^*_{-1} w'_{-1} = \mathbf{1}, \quad w'_{-1} w'^*_{-1} = e_1.$$

Moreover, as was noticed by R. Longo, [Lo2], we have the relation

$$J_{\mathcal{M}} w'_{-1} J_{\mathcal{N}} = w'_{-1}.$$

As a Corollary we get, see [Lo2],

Corollary 1 *Let $w_{-1} := \text{Ad}J_{\mathcal{N}}(w'_{-1}) \in \mathcal{N}$. Then w_{-1} is an isometry obeying*

$$w_{-1}^* w_{-1} = \mathbf{1}, \quad w_{-1} w_{-1}^* = e_{-1},$$

$$w_{-1} n = \gamma(n) w_{-1} \quad \forall n \in \mathcal{N}.$$

Proof : The first line follows immediately from the definitions. To prove the intertwiner property let $n \in \mathcal{N}$. Then

$$\begin{aligned} w_{-1} n &= J_{\mathcal{N}} v' J_{\mathcal{N}} n = J_{\mathcal{N}} (J_{\mathcal{M}} v' J_{\mathcal{N}}) J_{\mathcal{N}} n \\ &= J_{\mathcal{N}} J_{\mathcal{M}} v' n = J_{\mathcal{N}} J_{\mathcal{M}} n J_{\mathcal{M}} J_{\mathcal{N}} J_{\mathcal{N}} J_{\mathcal{M}} v' \\ &= \gamma(n) J_{\mathcal{N}} v' J_{\mathcal{N}} = \gamma(n) w_{-1} \end{aligned}$$

□

As an important tool we need the notion of a Pimsner-Popa basis of \mathcal{M} over \mathcal{N} via E_0 , see [Pi-Po]. Let $\{m_j\}_j \in \mathcal{M}$ be a family of elements in \mathcal{M} satisfying the conditions

1) $\{m_j e_1 m_j^*\}_j$ are mutually orthogonal projections

$$2) \quad \sum_j m_j e_1 m_j^* = 1.$$

Then the family $\{m_j\}_j \in \mathcal{M}$ is called a Pimsner-Popa basis associated with $E_0 : \mathcal{M} \rightarrow \mathcal{N}$. Using a Gram-Schmidt orthogonalization procedure one proves the existence of such a basis.

We need a special case of a Pimsner-Popa basis. Denote

$$\mu_0 := E_0|_{\mathcal{A}_{-1,0}} \rightarrow \mathbf{C}.$$

A state on $\mathcal{A}_{-1,0}$ can be viewed as a very special case of a conditional expectation, where the subalgebra of $\mathcal{A}_{-1,0}$ is $\mathbf{C}\mathbf{1} \subset \mathcal{A}_{-1,0}$. Therefore we may first construct a Pimsner-Popa basis associated with $(\mathcal{A}_{-1,0}, \mu_0)$.

Let \mathcal{H} be the GNS-Hilbert space associated with $(\mathcal{A}_{-1,0}, \mu_0)$. Using a Gram-Schmidt orthogonalization procedure we choose elements

$$\lambda_i \in \mathcal{A}_{-1,0} \subset \mathcal{M}$$

with

$$\mu_0(\lambda_i^* \lambda_j) = \delta_{ij},$$

$\lambda_0 = \mathbf{1}$, such that they yield an orthonormal basis in \mathcal{H} . For this special situation such a basis is a Pimsner-Popa basis, see [Pi-Po, He-Oc].

Now let the inclusion $\mathcal{N} \subset \mathcal{M}$ have depth 2, which says that $\mathcal{N}' \cap \mathcal{M}_2$ is a factor or

$$\mathbf{C} = \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}_1 \subset \mathcal{N}' \cap \mathcal{M}_2$$

is again a Jones tower, see [He-Oc, E-N]. Therefore we get

$$\sum_i \lambda_i e_1 \lambda_i^* = \mathbf{1}, \tag{2}$$

where the limit is taken in the strong topology on $\tilde{\mathcal{H}}$. Hence the above basis also gives a Pimsner-Popa basis for \mathcal{M} ,

$$\text{depth } 2 \Rightarrow \lambda_i \in \mathcal{A}_{-1,0} \subset \mathcal{M} \text{ is a Pimsner-Popa basis for } E_0 : \mathcal{M} \rightarrow \mathcal{N}.$$

Next we define shifted conditional expectations, Jones projections and intertwiners

$$E_{2n} := \gamma^{-n} \circ E \circ \gamma^{-n} : \mathcal{M}_{2n} \rightarrow \mathcal{M}_{2n-1}$$

$$e_{2n+1} := \gamma^{-n}(e_1) \in \mathcal{M}_{2n+1},$$

$$w_{2n-1} := \gamma^{-n}(w_{-1}) \in \mathcal{M}_{2n+1}.$$

Then

$$w_{2n+1}x = \gamma(x)w_{2n+1}, \quad \forall x \in \mathcal{M}_{2n+1}.$$

Our basic objects of interest are the relative commutants

$$\mathcal{A}_{i,j} := \mathcal{M}'_{i-1} \cap \mathcal{M}_j. \quad (3)$$

In particular we have

$$\mathcal{A}_{-1,0} := \mathcal{M} \cap \gamma(\mathcal{M}') = \mathcal{M} \cap \text{Ad } J_{\mathcal{N}}(\mathcal{M})$$

and

$$\mathcal{A}_{0,1} := \mathcal{M}_1 \cap \mathcal{N}' = \text{Ad } J_{\mathcal{M}}(\mathcal{N}') \cap \mathcal{N}'.$$

We will show below that in the irreducible depth 2 case $\mathcal{A}_{-1,0}$ carries a natural $*$ -Hopf-algebra structure with dual algebra $\mathcal{A}_{-2,-1}$. This can be generalized to all steps in the tower. The antipodes on $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ will turn out to be involutive, hence we are actually dealing with Kac algebras.

To identify the Haar state on $\mathcal{A}_{-1,0}$ we consider the restricted conditional expectations

$$E_{2n}|_{\mathcal{A}_{2n-1,2n}} \rightarrow \mathcal{A}_{2n-1,2n-1}.$$

Since we will assume irreducibility, i.e.

$$\mathcal{M} \cap \mathcal{N}' = \mathbf{C}.$$

we have

$$\gamma^{-n}(\mathcal{M} \cap \mathcal{N}') = \mathcal{A}_{2n,2n} = \mathbf{C}$$

and similarly $\mathcal{A}_{2n-1,2n-1} = \mathbf{C}$, and the restricted conditional expectations reduce to states. We will identify $E_0|_{\mathcal{A}_{-1,0}}$ to be the Haar state of $\mathcal{A}_{-1,0}$, when $\mathcal{A}_{-1,0}$ is equipped with the Hopf algebra structure mentioned above.

We close this section with a brief review of some useful identities, which hold for irreducible inclusions (not necessarily of depth 2) and are all due to R. Longo, [Lo2].

Lemma 2 (Lo2) $E_0 : \mathcal{M} \rightarrow \mathcal{N}$ is given by $m \mapsto w_{-1}^* \gamma(m) w_{-1}$.

Proof : One easily checks that $m \mapsto w_{-1}^* \gamma(m) w_{-1}$ defines a conditional expectation from $\mathcal{M} \rightarrow \mathcal{N}$. By irreducibility the conditional expectation is unique. \square

This later result leads to

Corollary 3

$$\mathbf{1}\mu_0(a) = w_{-1}^* \gamma(a) w_{-1} = w_1^* a w_1 = w_{-1}^* a w_{-1} \quad \forall a \in \mathcal{A}_{-1,0}.$$

Proof : For $a \in \mathcal{A}_{-1,0}$ we have $E_0(a) = \gamma^{-1} \circ E_0(a) \in \mathbf{C}$ and therefore

$$\mathbf{1}\mu_0(a) = w_{-1}^* \gamma(a) w_{-1} = \gamma^{-1}(w_{-1}^* \gamma(a) w_{-1}) = w_1^* a w_1.$$

To show that

$$w_{-1}^* a w_{-1} \in \mathcal{M} \cap \mathcal{N}' = \mathbf{C}.$$

let $m_{-1} \in \mathcal{N}$. Then

$$w_{-1}^* a w_{-1} m_{-1} = w_{-1}^* a \gamma(m_{-1}) w_{-1}.$$

But $\gamma(m_{-1}) \in \gamma(\mathcal{N})$ commutes with $a \in \mathcal{M} \cap \gamma(\mathcal{N}')$, and we get

$$= m_{-1} w_{-1}^* a w_{-1}.$$

Using the intertwining property $w_{-1} w_1 = w_1 w_{-1}$ we finally get

$$\begin{aligned} w_{-1}^* a w_{-1} &= w_1^* w_{-1}^* a w_{-1} w_1 = w_1^* w_1^* a w_1 w_1 \\ &= w_1^* a w_1 w_1^* w_1 = w_1^* a w_1. \end{aligned}$$

\square

We will frequently make use of the different descriptions of the state μ_0 . Sometimes we will implicitly identify $\mathbf{C}\mathbf{1} \subset \mathcal{A}_{-1,0} \cong \mathbf{C}$ and simply write $\mu_0(a) = w_{-1}^* a w_{-1}$ etc..

The last corollary we need is

Corollary 4 For $a \in \mathcal{A}_{-1,0}$ we have $w_1^* w_{-1}^* a w_1 = \mu_0(a) w_1^*$.

Proof : For $a \in \mathcal{A}_{-1,0}$ we have

$$w_1^* w_{-1}^* a w_1 = w_1^* w_1^* a w_1 = w_1^* \mu_0(a).$$

\square

3 The Fourier transform and the convolution product

To consider a special case let us assume we already knew that $\mathcal{L}(\mathcal{H}) \supset \mathcal{M} = \mathcal{N} \triangleleft G$ is a crossed product, where G is an abelian compact group acting outerly on \mathcal{N} . In this case the dual group \hat{G} would naturally act on \mathcal{M} and we would have a unique faithful conditional expectation from \mathcal{M} to \mathcal{N} given by the projection onto the \hat{G} -invariant part,

i.e. by averaging over \hat{G} w.r.t. the Haar measure on \hat{G} . G acts on \mathcal{N} and we can decompose \mathcal{N} into a direct sum of irreducible representations of G ,

$$\mathcal{N} = \bigoplus_{\hat{g} \in \hat{G}} \mathcal{N}_{\hat{g}}.$$

where for simplicity we have also assumed \hat{G} to be finite. Due to the assumption of \mathcal{N} being an infinite factor, we have a spatial isomorphism

$$\mathcal{N} \cong \mathcal{N}_{\hat{g}}.$$

Denote $u_{\hat{g}}$ the associated intertwiners. Again, according to the assumption of \mathcal{N} being an infinite factor, we have

$$\mathcal{N} \cong \mathcal{N}^{\# \hat{G}}.$$

It is not hard to see that

$$w_{-1} \cong \bigoplus_{\hat{g} \in \hat{G}} u_{\hat{g}}.$$

Moreover, by Landstad's result, see [Pe], we would have

$$(\mathcal{N} \triangleleft G \triangleleft \hat{G}) \cap \mathcal{N}' = L^\infty(G, \mu), \quad (\mathcal{N} \triangleleft G \triangleleft \hat{G} \triangleleft G) \cap \mathcal{M}' = L^1(G, \mu)$$

where $L^\infty(G, \mu)$ is the Fourier algebra, $L^1(G, \mu)$ the convolution algebra over G , μ the Haar state.

In terms of our Jones tower we would have

$$\mathcal{N} \triangleleft G \triangleleft \hat{G} \cong \mathcal{M}_1, \quad \mathcal{N} \triangleleft G \triangleleft \hat{G} \triangleleft G \cong \mathcal{M}_2.$$

Using the isomorphism γ we can reformulate the above observation as

$$\mathcal{A}_{-2,-1} = \gamma(\mathcal{N}' \cap \mathcal{M}_1) \cong L^\infty(G, \mu),$$

$$\mathcal{A}_{-1,0} = \gamma(\mathcal{M}' \cap \mathcal{M}_2) \cong L^1(G, \mu).$$

Moreover, by Takesaki duality, see [St], we would have

$$\mathcal{M}_2 \cong \mathcal{M} \otimes \mathcal{L}(L^2(G, \mu))$$

and under this identification the inclusion $\mathcal{N} \subset \mathcal{M}_2$ would become

$$\mathcal{N} \otimes \mathbf{1}_{L^2(G, \mu)} \subset \mathcal{M} \otimes \mathcal{L}(L^2(G, \mu)).$$

Hence $\mathcal{A}_{0,2} \equiv \mathcal{N}' \cap \mathcal{M}_2 = \mathcal{L}(L^2(G, \mu))$ which would imply Ocneanu's depth 2 condition. It was A. Ocneanu, see [Oc1,2], who first noticed that this picture generalizes to the case where G and \hat{G} are replaced by an arbitrary dual pair of finite dimensional Kac algebras. He also claimed that all irreducible finite index and depth 2 inclusions of factors arise in this way. Proofs of this statement have later been provided by [Lo, Szy, Da]. A rather extensive generalization to infinite index inclusions has recently been given by M. Enock and R. Nest, [E-N]. The aim in our paper is to give an alternative identification of $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ as a pair of dual Kac algebras, such that $\mathcal{A}_{i,i+1}$ naturally acts on \mathcal{M}_i and

$$\mathcal{M}_{i+1} = \mathcal{M}_i \triangleleft \mathcal{A}_{i,i+1}.$$

Let us start with some generalities. To simplify the computation we will use as a concrete realization of the GNS-Hilbert space \mathcal{H} the subspace

$$\mathcal{H} \cong \overline{\mathcal{A}_{-1,0} w_{-1} \Omega} \subset \tilde{\mathcal{H}}. \quad (4)$$

We will use Dirac's ket-vector notation to denote

$$|a\rangle := a w_{-1} \Omega, \quad a \in \mathcal{A}_{-1,0} \quad (5)$$

such that the left action of $\mathcal{A}_{-1,0}$ on \mathcal{H} reads

$$a|b\rangle = |ab\rangle, \quad a, b \in \mathcal{A}_{-1,0}.$$

By Corollary 3 we could also use $w_1 \Omega$ instead of $w_{-1} \Omega$. There is a simple relation between both realizations:

Lemma 5 1. *Ad $J_{\mathcal{N}}(\mathcal{A}_{-1,0}) = \mathcal{A}_{-1,0}$,*

2. *$J_{\mathcal{N}} w_1 \Omega = w_{-1} \Omega$.*

Proof : 1) follows by the definition, $\mathcal{A}_{-1,0} = \mathcal{M} \cap \text{Ad } J_{\mathcal{N}}(\mathcal{M})$.

For 2) notice that by definition we have

$$w_1 \Omega = \xi \in P^{\natural}(\mathcal{M}, \Omega)$$

and therefore

$$J_{\mathcal{N}} w_1 \Omega = J_{\mathcal{N}} J_{\mathcal{M}} w_1 \Omega = \gamma(w_1) \Omega = w_{-1} \Omega \quad (6)$$

□

Let us remark here that with λ_i also $\bar{\lambda}_i := J_{\mathcal{N}} \lambda_i J_{\mathcal{N}} \in \mathcal{A}_{-1,0}$ is a Pimsner-Popa basis:

$$\begin{aligned} \mu_0(\bar{\lambda}_j^* \bar{\lambda}_i) &= \langle w_1 \Omega, \bar{\lambda}_j^* \bar{\lambda}_i w_1 \Omega \rangle \\ &= \langle w_1 \Omega, J_{\mathcal{N}} \lambda_j^* \lambda_i J_{\mathcal{N}} w_1 \Omega \rangle = \langle w_{-1} \Omega, \lambda_i^* \lambda_j w_{-1} \Omega \rangle \\ &= \mu_0(\lambda_i^* \lambda_j) = \delta_{i,j}. \end{aligned}$$

Especially we get

$$\begin{aligned} \mathbf{1} &= \sum_i \bar{\lambda}_i e_1 \bar{\lambda}_i^* \\ &= J_{\mathcal{N}} \left(\sum_i \lambda_i J_{\mathcal{N}} e_1 J_{\mathcal{N}} \lambda_i^* \right) J_{\mathcal{N}}. \end{aligned}$$

Now we use $e_{-1} = \gamma(e_1)$ and $\text{Ad } J_{\mathcal{M}}(e_1) = e_1$, to get $e_{-1} = J_{\mathcal{N}} e_1 J_{\mathcal{N}}$, which yields

$$\sum_i \lambda_i e_{-1} \lambda_i^* = \mathbf{1}. \quad (7)$$

We also immediately get the

Corollary 6 *Assume depth 2. Then $\mathcal{A}_{-2,0} = \mathcal{M} \cap \gamma(\mathcal{N})'$ can naturally be identified with $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}_{-1,0} \subset \mathcal{A}_{-2,0}$ acts by the GNS-representation on \mathcal{H} .*

Proof : By the assumption of depth 2 we get

$$\mathbf{C} = \mathcal{A}_{-1,-1} \subset \mathcal{A}_{-1,0} \subset \mathcal{A}_{-1,1}.$$

is a Jones tower. This implies that

$$\{ae_1b \mid a, b \in \mathcal{A}_{-1,0}\} \subset \mathcal{A}_{-1,1}$$

is a dense $*$ -subalgebra, see [Pi-Po, He-Oc, E-N]. Now we use again $e_{-1} = J_{\mathcal{N}}e_1J_{\mathcal{N}}$ which yields

$$\{ae_{-1}b \mid a, b \in \mathcal{A}_{-1,0}\} \subset \text{Ad } J_{\mathcal{N}} (\mathcal{A}_{-1,1}) = \mathcal{A}_{-2,0} \quad (8)$$

is a dense $*$ -subalgebra. Define

$$\Lambda(ae_{-1}b^*) := |a \rangle \langle b| \in \mathcal{L}(\mathcal{H}),$$

where we used the bra-ket notion for vectors and forms. Λ can be continued to a normal morphism on $\mathcal{A}_{-2,0}$. For $a \in \mathcal{A}_{-1,0} \subset \mathcal{A}_{-2,0}$ we rewrite, using the Pimsner-Popa-basis λ_i ,

$$a = \sum_i a \lambda_i e_{-1} \lambda_i^*$$

which yields

$$\Lambda(a) = \sum_i |a \lambda_i \rangle \langle \lambda_i| = a \sum_i |\lambda_i \rangle \langle \lambda_i| = a$$

where we used that $|\lambda_i \rangle$ is an orthonormal basis of \mathcal{H} . This shows compatibility with the GNS-representation. \square

By definition we have

$$\mathcal{A}_{-2,-1} = \mathcal{N} \cap \gamma(\mathcal{N}') \subset \mathcal{A}_{-2,0} = \mathcal{L}(\mathcal{H})$$

and we identify $\mathcal{A}_{-1,0}$ resp. $\mathcal{A}_{-2,-1}$ with their images in $\mathcal{L}(\mathcal{H})$.

As a first step towards identifying $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ as a dual pair of Kac algebras we construct what will become the Fourier transform:

$$\begin{aligned} \mathcal{F} : \mathcal{A}_{-1,0} &\rightarrow \mathcal{A}_{-2,-1} \\ a &\mapsto \gamma(w_{-1}^* a w_1) = w_{-3}^* \gamma(a) w_{-1}. \end{aligned} \quad (9)$$

To see that $\mathcal{F}(a) \in \mathcal{A}_{-2,-1}$ first note that obviously $\mathcal{F}(a) \in \mathcal{N}$. Let $n \in \mathcal{N}$. Then

$$\begin{aligned} \gamma(w_{-1}^* a w_1) \gamma(n) &= \gamma(w_{-1}^* a w_1 n) \\ &= \gamma(w_{-1}^* a \gamma(n) w_1) \quad (\text{by using the intertwiner property of } w_1) \\ &= \gamma(w_{-1}^* \gamma(n) a w_1) \quad (\text{since } a \in \gamma(\mathcal{N})') \\ &= \gamma(n w_{-1}^* a w_1). \end{aligned} \quad (10)$$

Therefore $\mathcal{F}(a) = \gamma(n)\gamma(w_{-1}^*aw_1) \in \mathcal{N} \cap \gamma(\mathcal{N})' = \mathcal{A}_{-2,-1}$.

Moreover \mathcal{F} is injective, since

$$\gamma(w_{-1}^*aw_1) = 0 \Rightarrow w_{-1}w_{-1}^*aw_1w_1^* = e_{-1}ae_1 = 0 \Rightarrow e_{-1}a = 0,$$

where the last implication is standard from Jones theory, since $e_{-1}a \in \mathcal{M}$. Using $e_{-1} = J_N e_1 J_N$, see the proof of Corollary 6, and $J_N a J_N \in \mathcal{M}$ by the first identity in Lemma 5, the same argument also gives $a = 0$, i.e. \mathcal{F} is injective. We will show below, see Theorem 17, that \mathcal{F} has dense range.

Definition 7 *The injective linear map $\mathcal{F} : \mathcal{A}_{-1,0} \rightarrow \mathcal{A}_{-2,-1}$ is called the Fourier transform.*

We will show below that this map is really the Fourier transform of the underlying Kac-structure.

In the finite index case the notion of a Fourier transform was introduced by A. Ocneanu, see [Oc1,2]. A simple computation shows that in the finite index case his formula agrees with ours. It is at this point where the language of infinite algebras has the advantage of allowing a unified description which also works in the infinite index case.

We also have a natural convolution product $*$ on $\mathcal{A}_{-1,0}$:

$$\begin{aligned} * : \mathcal{A}_{-1,0} \times \mathcal{A}_{-1,0} &\rightarrow \mathcal{A}_{-1,0} \\ (a, b) &\mapsto a * b := w_{-1}^* a \gamma(b) w_{-1} \end{aligned} \quad (11)$$

To see that indeed $a * b \in \mathcal{A}_{-1,0}$, $a, b \in \mathcal{A}_{-1,0}$, let $m_{-2} \in \mathcal{M}_{-2} \equiv \gamma(\mathcal{M})$. Then

$$\begin{aligned} w_{-1}^* a \gamma(b) w_{-1} m_{-2} &= w_{-1}^* a \gamma(b) \gamma(m_{-2}) w_{-1} \quad (\text{by the intertwiner property}) \\ &= w_{-1}^* \gamma(m_{-2}) a \gamma(b) w_{-1} \quad (\text{since } a \gamma(b) \in \gamma(\mathcal{M}_{-2})') \\ &= m_{-2} w_{-1}^* a \gamma(b) w_{-1}. \end{aligned}$$

This proves $a * b \in \mathcal{M} \cap \gamma(\mathcal{M}') = \mathcal{A}_{-1,0}$.

Definition 8 *The product $* : \mathcal{A}_{-1,0} \times \mathcal{A}_{-1,0} \mapsto \mathcal{A}_{-1,0}$ is called the convolution product on $\mathcal{A}_{-1,0}$.*

In order to have consistent notations we check

Lemma 9

$$\mathcal{F}(a * b) = \mathcal{F}(a) \cdot \mathcal{F}(b) \quad \forall a, b \in \mathcal{A}_{-1,0}.$$

Proof : Using successively the intertwiner properties of the w_i 's we compute

$$\begin{aligned} \mathcal{F}(a * b) &= \gamma(w_{-1}^* w_{-1}^* a \gamma(b) w_{-1} w_1) \\ &= \gamma(w_{-1}^* w_{-3}^* a \gamma(b) w_1 w_1) \\ &= \gamma(w_{-1}^* a w_{-3}^* w_1 b w_1) \quad (\text{since } a \in \mathcal{M}'_{-2} \subset \mathcal{M}'_{-3}) \\ &= \gamma(w_{-1}^* a w_1 w_{-1}^* b w_1) = \mathcal{F}(a) \cdot \mathcal{F}(b) \end{aligned}$$

□

The injectivity of the Fourier transform implies

Corollary 10 *The convolution product is associative.*

Note that the formulae for the Fourier transform and convolution product also make sense if the underlying inclusion does not have depth 2, as was first pointed out by A. Ocneanu, [Oc1,2]. Moreover, even if the inclusion $\mathcal{N} \subset \mathcal{M}$ is not irreducible but of finite index there exists a unique minimal conditional expectation $E_0 : \mathcal{M} \rightarrow \mathcal{N}$, see [Ko,Lo]. Taking as above the uniquely associated intertwiner $w_{-1} \in \mathcal{N}$ one gets natural generalizations of the above notions to the reducible cases. A more detailed analysis of these generalizations is still under investigation, [N-W].

4 Coproducts, Haar weights and the Plancherel Formula

In this section we start with the definition of the coproduct

$$\begin{aligned} \Delta : D(\Delta) \subset \mathcal{A}_{-1,0} &\rightarrow \mathcal{A}_{-1,0} \otimes \mathcal{A}_{-1,0} \\ a &\mapsto a^{(1)} \otimes a^{(2)} \end{aligned}$$

where we used the Sweedler notation implying a summation convention in $\mathcal{A}_{-1,0} \otimes \mathcal{A}_{-1,0}$. The coproduct is defined as the $L^2(\mathcal{A}_{-1,0}, \mu_0)$ -transpose of the convolution product, i.e.

$$\mu_0(a(b * c)) =: (\mu_0 \otimes \mu_0)((a^{(1)} \otimes a^{(2)})(b \otimes c)), \quad \forall b, c \in \mathcal{A}_{-1,0}. \quad (12)$$

and $a \in D(\Delta)$ iff $\exists a^{(1)} \otimes a^{(2)} \in \mathcal{A}_{-1,0} \otimes \mathcal{A}_{-1,0}$ such that equation (12) holds. Here we have not assumed the depth 2 condition. Faithfulness of μ_0 implies that Δ is well defined and coassociative, since the convolution product $*$ is associative. In the finite index case, where we don't have to worry about domains of definitions, we get

Lemma 11

$$\begin{aligned} \Delta : \mathcal{A}_{-1,0} &\rightarrow \mathcal{A}_{-1,0} \otimes \mathcal{A}_{-1,0} \\ a &\mapsto \sum_{i,j} \lambda_i \otimes \lambda_j w_{-1}^* \gamma(\lambda_j^*) \mathcal{F}(\lambda_i)^* \gamma(a) w_{-1}. \end{aligned} \quad (13)$$

Proof : Let $a, b, c \in \mathcal{A}_{-1,0}$. Then

$$\begin{aligned} \mu_0(b * c, a) &= w_{-1}^* (w_{-1}^* b^* \gamma(c^*) w_{-1}) a w_{-1} \\ &= w_{-1}^* w_{-3}^* b^* \gamma(c^*) w_{-1} a w_{-1} \\ &= w_{-1}^* b^* w_{-3}^* \gamma(c^*) w_{-1} a w_{-1}. \end{aligned}$$

Now we use the equation (2) and Corollary 1 to conclude

$$\begin{aligned}
&= \sum_i w_{-1}^* b^* (\lambda_i w_{-1} w_{-1}^* \lambda_i^*) w_{-3}^* \gamma(c^*) w_{-1} a w_{-1} \\
&= \sum_i (w_{-1}^* b^* \lambda_i w_{-1}) w_{-1}^* w_{-3}^* \lambda_i^* \gamma(c^*) w_{-1} a w_{-1} \\
&= \sum_i \mu_0(b^* \lambda_i) \mu_0((\lambda_i^* * c^*) a).
\end{aligned}$$

We observe

$$\begin{aligned}
\mu_0((\lambda_i^* * c^*) a) &= w_1^* w_{-1}^* \lambda_i^* \gamma(c^*) w_{-1} a w_1 \\
&= w_1^* c^* w_1^* \lambda_i^* w_{-1} a w_1.
\end{aligned}$$

Now we use again that λ_j is a Pimsner-Popa basis for $E_0 : \mathcal{M} \rightarrow \mathcal{N}$,

$$\mathbf{1} = \sum_j \lambda_j e_1 \lambda_j^* = \sum_j \lambda_j w_1 w_1^* \lambda_j^* = \sum_j \lambda_j w_{-1}^* \gamma(\lambda_j^*) w_1 \quad (14)$$

and plugging this into the above formula yields

$$\begin{aligned}
\mu_0((\lambda_i^* * c^*) a) &= \sum_j w_1^* c^* \lambda_j w_{-1}^* \gamma(\lambda_j^*) w_1 w_1^* \lambda_i^* w_{-1} a w_1 \\
&= \sum_j w_1^* c^* (\lambda_j w_{-1}^* \gamma(\lambda_j^*) w_{-1}^* \gamma(\lambda_i^*) w_{-3} \gamma(a) w_{-1} w_1) \\
&= \mu_0(c^* (\sum_j \lambda_j w_{-1}^* \gamma(\lambda_j^*) \mathcal{F}(\lambda_i)^* \gamma(a)) w_{-1})
\end{aligned}$$

□

Under the assumption of depth 2 it will follow from section 6 that the formula for the coproduct is also valid in the infinite index case with $D(\Delta) = \mathcal{A}_{-1,0}$ and moreover that it is an algebra morphism, i.e. $\Delta(ab) = \Delta(a)\Delta(b)$, $a, b \in \mathcal{A}_{-1,0}$. Here we only note that $\mathbf{1} \in D(\Delta)$ and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and $\Delta(a)^* = \Delta(a^*)$, $a \in D(\Delta)$, which follows from

Lemma 12

$$\begin{aligned}
i) \quad &\mu_0(a * b) = \mu_0(a) \mu_0(b), \quad a, b \in \mathcal{A}_{-1,0} \\
ii) \quad &(a * b)^* = a^* * b^*, \quad a, b \in \mathcal{A}_{-1,0}
\end{aligned}$$

Proof : i) Using Corollary 3 together with $\mu_0(b) = \gamma(\mu_0(b)) \in \mathbf{C}\mathbf{1}$, $b \in \mathcal{A}_{-1,0}$, we have for $a, b \in \mathcal{A}_{-1,0}$

$$\begin{aligned}
\mu_0(a * b) &= w_{-1}^* w_{-1}^* a \gamma(b) w_{-1} w_{-1} = w_{-1}^* w_{-3}^* a \gamma(b) w_{-3} w_{-1} \\
&= w_{-1}^* a \gamma(w_{-1}^* b w_{-1}) w_{-1} = w_{-1}^* a \gamma(\mu_0(b)) w_{-1} \\
&= \mu_0(a) \mu_0(b).
\end{aligned}$$

ii) Follows immediately from the definition of the convolution product, since $a \gamma(b) = \gamma(b) a$, $a, b \in \mathcal{A}_{-1,0}$. □

Clearly i) implies $\mathbf{1} \in D(\Delta)$ and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, whereas $\Delta(a)^* = \Delta(a^*)$, $a \in D(\Delta)$, follows from ii), provided μ_0 is a trace, which has been shown by S. Yamagami, [Ya1], see also [Ya2]. In the remark to Proposition 19 below we state a conjecture, which also would imply the trace property of μ_0 and which in the finite index case is well known to hold.

Next we show that μ_0 is left and right invariant w.r.t. Δ and may hence be called a Haar state.

Lemma 13

$$(id \otimes \mu_0)(\Delta(a)) = (\mu_0 \otimes id)(\Delta(a)) = \mu_0(a)\mathbf{1}, \quad a \in D(\Delta).$$

Proof : From the definition of the convolution product and Corollary 3 we have for all $b \in \mathcal{A}_{-1,0}$

$$b * \mathbf{1} = \mathbf{1} * b = \mu_0(b)\mathbf{1}.$$

Hence, we have for $a \in D(\Delta)$, $b \in \mathcal{A}_{-1,0}$

$$(\mu_0 \otimes \mu_0)((\mathbf{1} \otimes b)\Delta(a)) = \mu_0((\mathbf{1} * b)a) = \mu_0(b)\mu_0(a),$$

$$(\mu_0 \otimes \mu_0)((b \otimes \mathbf{1})\Delta(a)) = \mu_0((b * \mathbf{1})a) = \mu_0(b)\mu_0(a),$$

from which the Lemma follows by faithfulness of μ_0 . □

We postpone the construction of the antipode on $\mathcal{A}_{-1,0}$ to a later stage, see Definition 18 below.

Next we remark that by Lemma 9 $\mathcal{A}_{-2,-1}$ naturally appears as the dual algebra of $(\mathcal{A}_{-1,0}, \Delta)$, where the pairing is given by

$$\langle a, \mathcal{F}(b) \rangle := \mu_0(ab), \quad a, b \in \mathcal{A}_{-1,0}.$$

In particular, since $\mathcal{F}(\mathbf{1}) = e_{-1}$, we get

$$\mu_0(a) = \langle a, e_{-1} \rangle,$$

and hence e_{-1} is the two-sided integral in $\mathcal{A}_{-2,-1}$. Indeed, we also have a counit $\hat{\varepsilon}$ on $\mathcal{A}_{-2,-1}$ given on $\hat{a} \in \mathcal{A}_{-2,-1}$ by

$$\hat{\varepsilon}(\hat{a}) = w_{-1}^* \hat{a} w_{-1} \in \mathcal{A}_{-1,-1} = \mathbf{C}.$$

Using equation (9) and Corollary 3 one easily checks that

$$\hat{\varepsilon}(\mathcal{F}(b)) = \langle \mathbf{1}, \mathcal{F}(b) \rangle = \mu_0(b),$$

and Lemma 13 implies for all $\hat{a} \in \mathcal{A}_{-2,-1}$

$$\hat{a}e_{-1} = e_{-1}\hat{a} = \hat{\varepsilon}(\hat{a})e_{-1}.$$

We now construct the Haar weight on the dual algebra $\mathcal{A}_{-2,-1} \cong \mathcal{A}_{0,1}$. Let us assume finite index for the moment. Then we can treat the inclusion $\mathcal{M} \subset \mathcal{M}_1$ on the same footing as $\mathcal{N} \subset \mathcal{M}$ and define an intertwiner $w_0 \in \mathcal{M}$ (and hence $w_{2n} \in \mathcal{M}_{2n}$) analogously as w_{-1} (and hence w_{2n-1}). We just have to start from the conditional expectation $E_1 : \mathcal{M}_1 \rightarrow \mathcal{M}$ associated to $E_0 : \mathcal{M} \rightarrow \mathcal{N}$ via Jones theory. In particular we get

$$w_0 m = \gamma(m) w_0, \quad m \in \mathcal{M} \quad (15)$$

and the normalization of w_0 could be fixed by requiring

$$w_{-1}^* w_0 = w_1^* w_0 = \mathbf{1}, \quad (16)$$

see [Lo1], where the above structure has been called an “irreducible Q-system”. Inspired by Lemma 2 we then get a state μ_1 on $\mathcal{A}_{0,1} \cong \mathcal{A}_{-2,-1}$ by

$$\mu_1(a) = w_0^* \gamma(a) w_0, \quad a \in \mathcal{A}_{0,1}$$

which apart from the normalization coincides with the restriction $E_1|_{\mathcal{A}_{0,1}}$.

We now show how these ideas carry over to the infinite index case. To this end we will derive formulas for w_0 and E_1 for the finite index case and show that the resulting formula for μ_1 also makes sense in the infinite index case.

Using the depth 2 condition we know that $\lambda_i \in \mathcal{A}_{-1,0} \subset \mathcal{M}$ is also a Pimsner-Popa basis for $E_0 : \mathcal{M} \rightarrow \mathcal{N}$. Hence

$$\mathbf{1} = \sum_i \lambda_i e_1 \lambda_i^* \in \mathcal{M}_1.$$

Using $e_1 = w_1 w_1^*$ and $m w_1^* = w_1^* \gamma(m)$, $m \in \mathcal{M}$, we get

$$= w_1^* \left(\sum_i \gamma(\lambda_i) w_{-1} \lambda_i^* \right),$$

an identity which we already used in the proof of Lemma 11. Hence we are led to the Ansatz

$$w_0 = \sum_i \gamma(\lambda_i) w_{-1} \lambda_i^*,$$

which is obviously well defined in the finite index case. With this Ansatz we compute

$$\begin{aligned} w_{-1}^* w_0 &= \sum_i w_{-1}^* \gamma(\lambda_i) w_{-1} \lambda_i^* \\ &= \sum_i E_0(\lambda_i) \lambda_i^* = \mathbf{1}, \end{aligned}$$

where we have used Lemma 2. This proves (16).

To check (15) let $m \in \mathcal{M}$. Then

$$\begin{aligned} w_0 m &= \sum_i \gamma(\lambda_i) w_{-1} \lambda_i^* m \\ &= \sum_{i,j} \gamma(\lambda_i) w_{-1} E_0(\lambda_i^* m \lambda_j) \lambda_j^* \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j} \gamma(\lambda_i) w_{-1} w_{-1}^* \gamma(\lambda_i^* m \lambda_j) w_{-1} \lambda_j^* \\
&= \sum_i \gamma(\lambda_i e_1 \lambda_i^*) \gamma(m) \sum_j \gamma(\lambda_j) w_{-1} \lambda_j^* \\
&= \gamma(m) w_0,
\end{aligned}$$

where we have used that λ_i is a Pimsner-Popa basis for E_0 in the second line and Lemma 2 in the third line. This proves (15).

Motivated by Lemma 2 and the above computation we now define

Definition 14

$$\begin{aligned}
E_1 : \mathcal{M}_1^+ &\rightarrow \bar{\mathcal{M}}^+ \\
a &\mapsto w_0^* \gamma(a) w_0 = \sum_{i,j} \lambda_i w_{-1}^* \gamma(\lambda_i^* a \lambda_j) w_{-1} \lambda_j^*
\end{aligned} \tag{17}$$

Then we get in the finite index case:

Lemma 15 *Assume the depth 2 condition. Then E_1 is a normal faithful conditional expectation, obeying*

$$E_1(m e_1 m^*) = m m^* \quad m \in \mathcal{M}$$

Proof : Let $m = \sum_{l=1}^n \lambda_l n_l \in \mathcal{M}$, $n_l \in \mathcal{N}$. Then

$$m e_1 m^* \in \mathcal{M}_1^+,$$

and we compute

$$\begin{aligned}
E_1(m e_1 m^*) &= \sum_{i,j,l,l'} \lambda_i w_{-1}^* \gamma(\lambda_i^* \lambda_l n_l e_1 n_{l'}^* \lambda_{l'} \lambda_j) w_{-1} \lambda_j^* \\
&= \sum_{i,j,l,l'} \lambda_i \gamma(w_1^* \lambda_i^* \lambda_l n_l w_1 w_1^* n_{l'}^* \lambda_{l'} \lambda_j w_1) \lambda_j^*.
\end{aligned}$$

Using the intertwiner property, Corollary 3 and λ_i being a Pimsner-Popa basis we get

$$\begin{aligned}
&= \sum_{i,j,l,l'} \lambda_i \gamma(w_1^* \lambda_i^* \lambda_l w_1) n_l n_{l'}^* \gamma(w_1^* \lambda_{l'}^* \lambda_j w_1) \lambda_j^* \\
&= \sum_{l,l'} \lambda_l n_l n_{l'}^* \lambda_{l'}^* = m m^*.
\end{aligned} \tag{18}$$

Now the span of $\{m e_1 \hat{m} \in \mathcal{M}_1 | m, \hat{m} \in \mathcal{M}\}$ is \mathcal{M}_1 and the proof is finished. \square

Note that irreducibility implies uniqueness of such a conditional expectation.

Remark It is easy to see that in the infinite index case E_1 is also well defined as an operator valued weight. Normality follows easily by the above formulas:

Let

$$\begin{aligned}
E_1^{I(N)} : \mathcal{M}_1^+ &\rightarrow \bar{\mathcal{M}}^+ \\
a &\mapsto \sum_{i,j \in I(N)} \lambda_i w_{-1}^* \gamma(\lambda_i^* a \lambda_j) w_{-1} \lambda_j^*
\end{aligned}$$

with $I^{(N)} \subset I$ a finite subset. Then $E_1^{I^{(N)}}$ is obviously normal and therefore also

$$E_1 := \overline{\lim_{I^{(N)} \subset I} E_1^{I^{(N)}}}.$$

On finite linear combinations $m = \sum_{l=1}^n \lambda_l n_l$ with $n_l \in \mathcal{N}$ the proof of Lemma 15 also applies to the above limit, showing $E_1(me_1 m^*) = mm^*$. Such finite linear combinations are dense in \mathcal{M} and we get semifiniteness of E_1 . Conversely one could take the property

$$E_1(me_1 m^*) = mm^* \quad m \in \mathcal{M}$$

as the starting point of a definition, as was first done in [B-D-H].

In our analysis we don't need the specific properties of the operator valued weight, so we will skip these points. We now show how the definition of $\mu_{-1} = E_1 \circ \gamma^{-1}|_{\mathcal{A}_{-2,-1}}$ generalizes to the infinite index case. To this end we use the following identities, valid in the finite index case for all $x \in \mathcal{A}_{-2,-1}$, similarly as in Corollary 3:

$$\begin{aligned} \mu_{-1} : \mathcal{A}_{-2,-1}^+ &\rightarrow \bar{\mathbf{R}}^+ \\ x^* x &\mapsto E_1(\gamma^{-1}(x^* x)) = w_0^*(x^* x)w_0 \\ &= w_0^*(\gamma^{-1}(x^* x))w_0 \\ &= \sum_{i,j} \lambda_i w_{-1}^* \gamma(\lambda_i^*) \gamma^{-1}(x^* x) \gamma(\lambda_j) w_{-1} \lambda_j^*. \end{aligned}$$

Now $\gamma^{-1}(x^* x) \in \mathcal{A}_{0,1} \subset \mathcal{N}'$ and $\gamma(\lambda_i) \in \gamma(\mathcal{M}) \subset \mathcal{N}$ and we conclude

$$\mu_{-1}(x^* x) = \sum_{i,j} \lambda_i w_{-1}^* \gamma(\lambda_i^*) \gamma(\lambda_j) w_{-1} \gamma^{-1}(x^* x) \lambda_j^* = \sum_i \lambda_i \gamma^{-1}(x^* x) \lambda_i^*. \quad (19)$$

The last expression is the important one, since it also makes sense in the infinite index case.

Definition 16 On $\mathcal{A}_{-2,-1}$ let μ_{-1} be the operator valued weight

$$\begin{aligned} \mu_{-1} : \mathcal{A}_{-2,-1}^+ &\rightarrow \mathcal{A}_{-2,-1}^{+} \\ x^* x &\mapsto \sum_i \lambda_i \gamma^{-1}(x^* x) \lambda_i^*. \end{aligned}$$

We now show that μ_{-1} is in fact a good Ansatz for the Haar weight on $\mathcal{A}_{-2,-1}$, since we have the following Plancherel formula:

Theorem 17 If the depth of $\mathcal{N} \subset \mathcal{M}$ is two, then μ_{-1} defines a semifinite faithful normal weight and the Fourier transform extends to a unitary

$$\mathcal{F} : L^2(\mathcal{A}_{-1,0}, \mu_0) \rightarrow L^2(\mathcal{A}_{-2,-1}, \mu_{-1}),$$

i.e. \mathcal{F} has dense range and

$$\mu_{-1}(\mathcal{F}(a)^* \mathcal{F}(a)) = \mu_0(a^* a) \quad \forall a \in \mathcal{A}_{-1,0}.$$

Proof : Let $a \in \mathcal{A}_{-1,0}$. Then

$$\begin{aligned}\mu_{-1}(\mathcal{F}(a)^*\mathcal{F}(a)) &= \sum_i \lambda_i w_1^* a^* w_{-1} w_{-1}^* a w_1 \lambda_i^* \\ &= \sum_i \lambda_i w_1^* a^* e_{-1} a w_1 \lambda_i^* \\ &= \sum_i w_1^* \gamma(\lambda_i) a^* e_{-1} a \gamma(\lambda_i^*) w_1\end{aligned}$$

by using the intertwiner property, and from $\gamma(\lambda_i) \in \gamma(\mathcal{M}) \subset \mathcal{N}$, $a \in \mathcal{N}'$ we conclude

$$\begin{aligned}&= \sum_i w_1^* a^* \gamma(\lambda_i) e_{-1} \gamma(\lambda_i^*) a w_1 \\ &= w_1^* a^* a w_1 = \mu_0(a^* a)\end{aligned}$$

where we used λ_i a Pimsner-Popa basis. The Plancherel formula generalizes obviously to $\mu_{-1}(\mathcal{F}(a)^*\mathcal{F}(b)) = \mu_0(a^* b)$, $\forall a, b \in \mathcal{A}_{-1,0}$.

Next we show that $\{\mathcal{F}(a)^*\mathcal{F}(b) | a, b \in \mathcal{A}_{-1,0}\} \subset \mathcal{A}_{-2,-1}$ is dense. To this end we compute

$$\begin{aligned}\mathcal{F}(a)^*\mathcal{F}(b) &= \gamma(w_1^* a^* w_{-1} w_{-1}^* b w_1) = \gamma(w_1^* a^* e_{-1} b w_1) \\ &= E_0(a^* e_{-1} b), \quad a, b \in \mathcal{A}_{-1,0}.\end{aligned}\tag{20}$$

From the proof of Corollary 6 we know

$$\{a e_{-1} b | a, b \in \mathcal{A}_{-1,0}\} \subset \mathcal{A}_{-2,0} \text{ is dense.}$$

The faithfulness of E_2 then proves

$$\{E_0(a e_{-1} b) | a, b \in \mathcal{A}_{-1,0}\} \subset \mathcal{A}_{-2,-1} \text{ is dense.}$$

Hence

$$\begin{aligned}\mu_{-1} : \mathcal{A}_{-2,-1}^+ &\rightarrow \bar{\mathbf{R}}^+ \cong \bar{\mathbf{R}}^+ \mathbf{1} \subset \mathcal{A}_{-2,-1}^+ \\ x^* x &\mapsto \sum_i \lambda_i \gamma^{-1}(x^* x) \lambda_i^*\end{aligned}$$

even defines a semifinite faithful normal weight. To prove that \mathcal{F} has dense range let $a, b \in \mathcal{A}_{-1,0}$. Then the Plancherel formula, Theorem 17 and the intertwiner property of w_1 show

$$\begin{aligned}\mu_{-1}(\mathcal{F}(\lambda_i)^*\mathcal{F}(a)^*\mathcal{F}(b)) &= \mu_{-1}((\mathcal{F}(a)\mathcal{F}(\lambda_i))^*\mathcal{F}(b)) = \mu_{-1}(\mathcal{F}(a * \lambda_i))^*\mathcal{F}(b) \\ &= \mu_0(a * \lambda_i)^* b = w_1^* w_1^* a * \gamma(\lambda_i)^* w_{-1} b w_1 \\ &= w_1^* \lambda_i^* w_1^* a^* w_{-1} b w_1.\end{aligned}$$

Therefore we get

$$\begin{aligned}\sum_i \mu_{-1}(\mathcal{F}(\lambda_i)^*\mathcal{F}(a)^*\mathcal{F}(b))\mathcal{F}(\lambda_i) &= \sum_i \gamma(w_{-1}^* \lambda_i w_1) \gamma(w_1^* \lambda_i^* w_1^* a^* w_{-1} b w_1) \\ &= \gamma\left(\sum_i w_{-1}^* (\lambda_i w_1 w_1^* \lambda_i^*) w_1^* a^* w_{-1} b w_1\right) \\ &= \gamma(w_{-1}^* w_1^* a^* w_{-1} b w_1)\end{aligned}$$

Now $w_{-1}^* \in \mathcal{N}$ and $w_1^* a^* w_{-1} = \gamma^{-1}(\mathcal{F}(a)^*) \in \mathcal{A}_{0,1} \subset \mathcal{N}'$ and we conclude

$$= \gamma(w_1^* a^* w_{-1} w_{-1}^* b w_1) = \mathcal{F}(a)^* \mathcal{F}(b).$$

This shows that the Fourier transform has dense range and, therefore, the continuation to an operator $\mathcal{F} : L^2(\mathcal{A}_{-1,0}, \mu_0) \rightarrow L^2(\mathcal{A}_{-2,-1}, \mu_{-1})$ is a unitary. \square

For the rest of this paper we will assume the depth 2 condition, i.e. the Fourier transform has dense range. We are now in the position to define the antipode on $\mathcal{A}_{-2,-1}$. First we get for $a, b \in \mathcal{A}_{-1,0}$

$$\begin{aligned} \mathcal{F}(a)^* |b\rangle &= w_{-1}^* \gamma(a^*) w_{-3} b w_{-1} \Omega \\ &= w_{-1}^* \gamma(a^*) b w_{-1} w_{-1} \Omega = |b * a^*\rangle \end{aligned}$$

where we used $b \in \mathcal{M}'_{-2}, w_{-3} \in \mathcal{M}_{-3} \subset \mathcal{M}_{-2}$ and the intertwiner property of w_{-1} . Let $S_{\mathcal{A}_{-1,0}}$ be the Tomita operator of the GNS representation $(\mathcal{A}_{-1,0}, w_{-1} \Omega)$. Then we compute

$$\begin{aligned} S_{\mathcal{A}_{-1,0}} \mathcal{F}(a)^* S_{\mathcal{A}_{-1,0}} |b\rangle &= S_{\mathcal{A}_{-1,0}} \mathcal{F}(a)^* |b^*\rangle = S_{\mathcal{A}_{-1,0}} |b^* * a^*\rangle \\ &= |b * a\rangle = \mathcal{F}(a^*)^* |b\rangle. \end{aligned} \tag{21}$$

As was proven by S. Yamagami, [Ya1], see also the remark at the end of this section, μ_0 is a trace, i.e.

$$S_{\mathcal{A}_{-1,0}} = J_{\mathcal{A}_{-1,0}}.$$

Hence the Tomita operator is equal to the antiunitary modular conjugation, and we get

$$\text{Ad } J_{\mathcal{A}_{-1,0}}(\mathcal{A}_{-2,-1}) = \mathcal{A}_{-2,-1}.$$

Definition 18 *The map*

$$\begin{aligned} \hat{S} : \mathcal{A}_{-2,-1} &\rightarrow \mathcal{A}_{-2,-1} \\ \hat{a} &\mapsto \text{Ad } J_{\mathcal{A}_{-1,0}}(\hat{a}^*) \end{aligned}$$

is called the antipode on $\mathcal{A}_{-2,-1}$. The map

$$S := \mathcal{F}^{-1} \circ \hat{S} \circ \mathcal{F} : \mathcal{A}_{-1,0} \rightarrow \mathcal{A}_{-1,0}$$

is called the antipode on $\mathcal{A}_{-1,0}$.

Using methods of S. Baaj and G. Skandalis, [Ba-Sk], the results of section 6 imply that \hat{S} really gives the antipode on $\mathcal{A}_{-2,-1}$. It then follows from the theory of Kac algebras that the antipode on $\mathcal{A}_{-1,0}$ is given by $S := \mathcal{F}^{-1} \circ \hat{S} \circ \mathcal{F}$. Also note that equation (22) implies $\hat{S}(\mathcal{F}(a)) = \mathcal{F}(a^*)^*, a \in \mathcal{A}_{-1,0}$.

The following properties are nearly obvious.

Proposition 19 *Let $\hat{a}, \hat{b} \in \mathcal{A}_{-2,-1}$. Then*

i) $\hat{S}(\hat{a}\hat{b}) = \hat{S}(\hat{b})\hat{S}(\hat{a})$

ii) $\hat{S}(\hat{a}^*) = \hat{S}(\hat{a})^*$

iii) $\hat{S} \circ \hat{S} = id$

i.e. \hat{S} defines an involutive anti-automorphism on $\mathcal{A}_{-2,-1}$. Furthermore μ_{-1} is \hat{S} -invariant.

Proof : We only prove the \hat{S} -invariance of μ_{-1} . For this let $\hat{a} \in D(\mu_{-1})$,

$$\begin{aligned} \mu_{-1}(\hat{a}) &= \sum_i \lambda_i \hat{a} \lambda_i^* = \sum_i \langle \lambda_i^* | \hat{a} | \lambda_i^* \rangle \\ &= \sum_i \langle \lambda_i | J_{\mathcal{A}_{-1,0}} \hat{a}^* J_{\mathcal{A}_{-1,0}} | \lambda_i \rangle = \sum_i \langle \lambda_i | \hat{S}(\hat{a}^*) | \lambda_i \rangle. \end{aligned}$$

Now μ_0 is a trace and therefore $\lambda_i^* \in \mathcal{A}_{-1,0}$ is also a Pimsner-Popa basis, i.e.

$$= \mu_{-1}(\hat{S}(\hat{a})).$$

□

It is easy to show that Proposition 19 implies the same properties for S . Moreover, μ_0 is S -invariant.

Remark

We close this section with some remarks concerning the tracial properties of μ_0 and μ_{-1} . First note that similar to the proof of Theorem 17 one can show that

$$E_{-2} \circ E_{-1} \circ E_0 | \mathcal{M} \cap \gamma(\mathcal{N}') \rightarrow \bar{\mathbf{C}}^+$$

defines a faithful normal weight, where the operator valued weights are defined by liftings. Due to a general result of U. Haagerup, [Ha1,2], which one can check to apply for our case, there is a bijection between semifinite normal faithful weights on $\mathcal{M} \cap \gamma(\mathcal{N}')$ and semifinite faithful normal operator valued weights from $\mathcal{M} \rightarrow \gamma(\mathcal{N})$, see also [St]. The assumption of depth 2 of the underlying inclusion implies the factor property of $\mathcal{M} \cap \gamma(\mathcal{N}')$ and from Corollary 6 we know $\mathcal{M} \cap \gamma(\mathcal{N}') \cong \mathcal{L}(\mathcal{H})$, i.e. there is a unique faithful normal tracial weight on $\mathcal{M} \cap \gamma(\mathcal{N})'$. We conjecture that the associated operator valued weight from \mathcal{M} to $\gamma(\mathcal{N})$ is exactly $E_{-2} \circ E_{-1} \circ E_0$. From the boundedness of E_0 resp. E_{-2} we get

$$E_{-2} \circ E_{-1} \circ E_0 | \mathcal{N} \cap \gamma(\mathcal{N}') = \mu_{-1}$$

i.e. if the above conjecture holds, then μ_{-1} is tracial. We have been told by S. Yamagami that he has proven such a result, [Ya 1,2].

In the finite index case the conjecture is true by the following arguments. Irreducibility implies that the conditional expectations E_i are minimal. By the work of H. Kosaki and R. Longo, [Ko-Lo], the composition of minimal conditional expectations is again

minimal and we conclude that $E_{-2} \circ E_{-1} \circ E_0$ is minimal. But for minimal conditional expectations the restriction onto the relative commutant is tracial, see [Lo 2].

In any case we have the following

Lemma 20 μ_{-1} is tracial iff μ_0 is a trace.

Proof : Let μ_0 be a trace, λ_i a Pimsner-Popa basis as above. Then

$$\mu_0(\lambda_i \lambda_j^*) = \mu_0(\lambda_j^* \lambda_i) = \delta_{i,j},$$

i.e. λ_i^* is again a Pimsner-Popa basis. We get for $x \in \mathcal{A}_{-2,-1}$

$$\begin{aligned} \mu_{-1}(\gamma(x^*x)) &= \sum_i \lambda_i^* x^* x \lambda_i = \sum_i \lambda_i x^* x \lambda_i^* \\ &= w_{-1}^* \sum_i \lambda_i^* x^* x \lambda_i w_{-1} = \sum_i \langle \lambda_i | x^* x | \lambda_i \rangle \\ &= \text{tr}_{\mathcal{H}} x^* x. \end{aligned} \tag{22}$$

This also proves μ_{-1} tracial.

To prove the opposite direction let us note a general result. Define

$$\begin{aligned} \hat{\mu}_0 : \mathcal{A}_{-1,0}^+ &\rightarrow \overline{\mathcal{A}_{-3,-1}}^+ \\ a &\mapsto \sum \mathcal{F}(\lambda_i) \gamma(a) \mathcal{F}(\lambda_i)^*. \end{aligned}$$

Then we compute for $a \in \mathcal{A}_{-1,0}^+$

$$\begin{aligned} \hat{\mu}_0(a) &= \gamma\left(\sum_i w_{-1}^* \lambda_i w_1 a w_1^* \lambda_i^* w_{-1}\right) \\ &= \gamma\left(\sum_i w_{-1}^* \gamma(a) \lambda_i w_1 w_1^* \lambda_i^* w_{-1}\right) \end{aligned}$$

where we used the intertwiner property of w_1 and $\lambda_i \in \gamma(\mathcal{M})'$. From λ_i being a Pimsner-Popa basis for $E_0 : \mathcal{M} \rightarrow \mathcal{N}$ we conclude

$$= \gamma(w_{-1}^* \gamma(a) w_{-1}) = \mu_0(a).$$

Using the Plancherel formula, Theorem 17, we get

$$\mathcal{F}(\lambda_i) \in \mathcal{A}_{-2,-1} \quad \text{is an orthonormal basis in } L^2(\mathcal{A}_{-2,-1}, \mu_{-1}).$$

Now the definition of $\hat{\mu}_0$ does not depend on the special choice of such a basis and we can argue as before, i.e. if μ_{-1} is tracial, we get μ_0 is a trace. □

5 The Crossed Product

In this section we identify

$$\mathcal{A}_{0,1} \cong \mathcal{A}_{-2,-1} = \gamma(\mathcal{A}_{0,1})$$

and put

$$\mathcal{F}_1 := \gamma^{-1} \circ \mathcal{F}, \quad \hat{S}_1 := \gamma^{-1} \circ \hat{S} \circ \gamma,$$

and

$$\hat{\varepsilon}_1 = \hat{\varepsilon} \circ \gamma.$$

We then extend the natural right action of the compact Kac algebra $\mathcal{A}_{0,1}$ on its dual $\mathcal{A}_{-1,0} \subset \mathcal{M}$ to an action on the whole of \mathcal{M} , such that $\mathcal{N} \subset \mathcal{M}$ is given as the $\mathcal{A}_{0,1}$ invariant subalgebra and the conditional expectation $E_0 : \mathcal{M} \rightarrow \mathcal{N}$ is given as the right action of the integral (\equiv Haar element) $e_1 \in \mathcal{A}_{0,1}$. We then prove that \mathcal{M}_1 is in fact the crossed product

$$\mathcal{M}_1 = \mathcal{M} \ltimes \mathcal{A}_{0,1}.$$

Throughout this section we assume the depth 2 condition on $\mathcal{N} \subset \mathcal{M}$.

Definition 21 For $a \in \mathcal{A}_{-1,0}$ and $m \in \mathcal{M}$ we put

$$a * m := w_{-1}^* a \gamma(m) w_{-1}.$$

Note that for $m \in \mathcal{A}_{-1,0} \subset \mathcal{M}$ this coincides with our previous convolution product, equ. (11). Moreover we have

Proposition 22 Let $a, b \in \mathcal{A}_{-1,0}$ and $m \in \mathcal{M}$.

$$i) \quad (a * b) * m = a * (b * m)$$

$$ii) \quad \mathbf{1} * m = E_0(m)$$

iii) The following conditions a) - d) are equivalent

$$a) \quad n \in \mathcal{N}$$

$$b) \quad a * n = \mu_0(a)n \quad \forall a \in \mathcal{A}_{-1,0}$$

$$c) \quad a * (nm) = n(a * m), \quad \forall a \in \mathcal{A}_{-1,0}, m \in \mathcal{M}$$

$$d) \quad a * (mn) = (a * m)n, \quad \forall a \in \mathcal{A}_{-1,0}, m \in \mathcal{M}$$

Proof :

i) is straight forward from the definitions.

ii) $\mathbf{1} * m = w_{-1}^* \gamma(m) w_{-1} = E_0(m)$ by Lemma 2.

iii) Let us show the equivalence. a) \Rightarrow b) : $a * n = w_{-1}^* a \gamma(n) w_{-1} = w_{-1} a w_{-1} n = \mu_0(a)n$ where we used the intertwiner property of w_{-1} and Corollary 3.

a) \Rightarrow c) : $a * (nm) = w_{-1}^* \gamma(nm) a w_{-1} = n w_{-1}^* \gamma(m) a w_{-1} = n(a * m)$.

a) \Rightarrow d) is similarly proved as a) \Rightarrow c), using $\gamma(n)a = a\gamma(n)$.

b) \Rightarrow a) : Put $a = \mathbf{1}$ and apply ii)
c) \Rightarrow b) and d) \Rightarrow b) : Put $m = \mathbf{1}$ and note that $a * \mathbf{1} = w_{-1}^* a w_{-1} = \mu_0(a)$, see Corollary 3. \square

We now use Proposition 22 to construct a right Hopf-module action of $\mathcal{A}_{0,1}$ on \mathcal{M} which extends the natural right action of $\mathcal{A}_{0,1}$ on its dual $\mathcal{A}_{-1,0} \subset \mathcal{M}$ and which becomes an inner action in \mathcal{M}_1 . In order to avoid analytical problems at this point, let us assume for the moment finite index. It will follow from the results of section 6 that the formulas also hold in the infinite index case.

Definition 23 For $\hat{a} \in \mathcal{A}_{0,1}$ and $m \in \mathcal{M}$ denote

$$m \triangleleft \hat{a} := \mathcal{F}_1^{-1}(\hat{S}_1(\hat{a})) * m \in \mathcal{M}$$

This will be our right action. To have explicit formulas we put $w_{2n} := \gamma^{-n}(w_0)$ and compute for $\hat{a} \in \mathcal{A}_{0,1}, m \in \mathcal{M}$

$$\begin{aligned} \mathcal{F}_1^{-1}(\hat{a}) &= w_0^* \gamma(\hat{a}) w_{-2} \\ \mathcal{F}_1^{-1}(\hat{S}_1(\hat{a})) &= \mathcal{F}_1^{-1}(\hat{a}^*)^* = w_{-2}^* \gamma(\hat{a}) w_0 \in \mathcal{A}_{-1,0} \end{aligned}$$

and therefore

$$\begin{aligned} m \triangleleft \hat{a} &= w_{-1}^* \mathcal{F}_1^{-1}(\hat{S}_1(\hat{a})) \gamma(m) w_{-1} \\ &= w_{-1}^* \gamma(m) \mathcal{F}_1^{-1}(\hat{S}_1(\hat{a})) w_{-1} \\ &= w_{-1}^* \gamma(m) w_{-2}^* \gamma(\hat{a}) w_0 w_{-1}. \end{aligned} \tag{23}$$

Now we use the intertwiner property (15) for w_0 , which together with $\gamma(\hat{a}) \in \mathcal{A}_{-2,-1} \subset \mathcal{M}'_3$ yields

$$\begin{aligned} &= w_{-1}^* \gamma(m) w_{-2}^* w_{-3} \gamma(\hat{a}) w_0 \\ &= w_{-1}^* \gamma(m) \gamma(\hat{a}) w_0 \end{aligned} \tag{24}$$

where in the last line we used equation (16).

This later formula also makes sense in the infinite index case.

To prepare Theorem 26 below we need the following two Lemmas.

Lemma 24 For $m \in \mathcal{M}$ and $a \in \mathcal{A}_{-1,0}$ we have

$$(m \triangleleft \mathcal{F}_1(a))^* = m^* \triangleleft \mathcal{F}_1(a^*)$$

Proof : Using the antipode on $\mathcal{A}_{-1,0}$, $S = \mathcal{F}_1^{-1} \circ \hat{S}_1 \circ \mathcal{F}_1$, and Lemma 12 we have

$$\begin{aligned} (m \triangleleft \mathcal{F}_1(a))^* &= (S(a) * m)^* = S(a)^* * m^* \\ &= m^* \triangleleft \mathcal{F}_1(a)^*. \end{aligned}$$

Here we used that S commutes with $*$, which follows from Proposition 19 ii),iii) and the identity $\mathcal{F}_1(a^*) = \hat{S}_1(\mathcal{F}_1(a))^*$ implying

$$\begin{aligned} * \circ S \circ * &= (\mathcal{F}_1 \circ *)^{-1} \circ \hat{S}_1 \circ (\mathcal{F}_1 \circ *) = (* \circ \hat{S}_1 \circ \mathcal{F}_1)^{-1} \circ \hat{S}_1 \circ (* \circ \hat{S}_1 \circ \mathcal{F}_1) \\ &= \mathcal{F}_1^{-1} \circ \hat{S}_1^3 \circ \mathcal{F}_1 = S \end{aligned}$$

□

Lemma 25 For $a, b \in \mathcal{A}_{-1,0}$ we have

$$\mathcal{F}_1(b \triangleleft \mathcal{F}_1(a)) = \mathcal{F}_1(a^*)^* \mathcal{F}_1(b)$$

Proof : As in the above proof we compute

$$\begin{aligned} \mathcal{F}_1(b \triangleleft \mathcal{F}_1(a)) &= \mathcal{F}_1(S(a) * b) = \mathcal{F}_1(S(a)) \mathcal{F}_1(b) \\ &= \hat{S}_1(\mathcal{F}_1(a)) \mathcal{F}_1(b) = \mathcal{F}_1(a^*)^* \mathcal{F}_1(b). \end{aligned}$$

□

Now we are in the position to formulate

Theorem 26 For $\hat{a}, \hat{b} \in \mathcal{A}_{0,1}$ and $m \in \mathcal{M}$ we have

- i) $(m \triangleleft \hat{a}) \triangleleft \hat{b} = m \triangleleft (\hat{a}\hat{b})$
- ii) $m \triangleleft e_1 = E_0(m)$
- iii) $m \in \mathcal{N} \Leftrightarrow m \triangleleft \hat{a} = \hat{\varepsilon}_1(\hat{a})m \quad \forall \hat{a} \in \mathcal{A}_{0,1}$
- iv) $m \in \mathcal{A}_{-1,0} \Rightarrow m \triangleleft \hat{a} = \langle m^{(1)}, \gamma(\hat{a}) \rangle m^{(2)}$
- v) In \mathcal{M}_1 we have

$$m \triangleleft \hat{a} = \hat{S}_1(\hat{a}^{(1)})m\hat{a}^{(2)}$$

where $\hat{a}^{(1)} \otimes \hat{a}^{(2)} = \hat{\Delta}(\hat{a})$ is the natural coproduct on $\mathcal{A}_{0,1}$ obtained from the pairing with $\mathcal{A}_{-1,0}$.

Proof : i) follows from Proposition 22 i), Proposition 19 i) and Lemma 9.

ii) follows from Proposition 22 ii), since $\mathcal{F}_1(\mathbf{1}) = e_1$.

iii) follows from Proposition 22 iii), since $\mu_0(a) = \hat{\varepsilon}_1(\mathcal{F}_1(a))$.

iv) Let $\hat{a} = \mathcal{F}_1(a)$ and $\hat{b} = \mathcal{F}_1(b)$. Then iv) is equivalent to

$$\langle (m \triangleleft \hat{a}), \gamma(\hat{b}) \rangle = \langle m, \gamma(\hat{a}\hat{b}) \rangle \equiv \mu_0(m(a * b)), \quad \forall m \in \mathcal{M}, a, b \in \mathcal{A}_{-1,0} \quad (25)$$

To prove (25) we use the Lemmas 25 and 26 and the Plancherel formula, Theorem 17, to get

$$\begin{aligned}
\langle (m \triangleleft \hat{a}), \gamma(\hat{b}) \rangle &= \mu_0((m \triangleleft \hat{a})b) \\
&= \mu_1(\mathcal{F}_1(m^* \triangleleft \mathcal{F}_1(a^*))^* \hat{b}) \\
&= \mu_1(\mathcal{F}_1(m^*)^* \mathcal{F}_1(a) \mathcal{F}_1(b)) = \mu_0(m(a * b)).
\end{aligned}$$

v) To prove v) we use the identity (24), $m \triangleleft \hat{a} = w_{-1}^* \gamma(m \hat{a}) w_0$, and the fact that also $S(\lambda_i)$ is a Pimsner-Popa basis, (since S extends to an unitary), to compute

$$\begin{aligned}
m \triangleleft \hat{a} &= \sum_i w_{-1}^* S(\lambda_i) w_1 w_1^* S(\lambda_i)^* \gamma(m) \gamma(\hat{a}) w_0 \\
&= \sum_i \mathcal{F}_1(S(\lambda_i)) m (S(\lambda_i) w_1)^* \gamma(\hat{a}) w_0 \\
&= \sum_i \hat{S}_1(\mathcal{F}_1(\lambda_i)) m w_0^* \mathcal{F}_1(\lambda_i^*) \gamma(\hat{a}) w_0
\end{aligned}$$

where we have used $S = \mathcal{F}_1^{-1} \circ \hat{S}_1 \circ \mathcal{F}_1$ and therefore

$$S(\lambda_i) w_1 = \mathcal{F}_1^{-1}(\mathcal{F}_1(\lambda_i^*)^*) w_1 = \mathcal{F}_1(\lambda_i^*)^* w_0.$$

Now $\mathcal{F}_1(\lambda_i^*) \in \mathcal{A}_{0,1}$ commutes with $\gamma(\hat{a}) \in \mathcal{A}_{-2,-1} \subset \mathcal{N}$ and $\mathcal{F}_1(\lambda_i^*) w_0 = \lambda_i^* w_1$. Hence v) holds, provided

$$\hat{\Delta}(\hat{a}) = \sum_i \mathcal{F}_1(\lambda_i) \otimes w_0^* \gamma(\hat{a}) \lambda_i^* w_1.$$

To check this formula for $\hat{\Delta}$ we have to use the pairing between $\mathcal{A}_{-1,0}$ and $\mathcal{A}_{-2,-1}$ and verify that $\hat{\Delta}(\hat{a})$ is the solution of

$$\mu_0(bc \mathcal{F}_1^{-1}(\hat{a})) = (\mu_0 \otimes \mu_0)((b \otimes c)(\mathcal{F}_1^{-1} \otimes \mathcal{F}_1^{-1})(\hat{\Delta}(\hat{a}))) \quad \forall b, c \in \mathcal{A}_{-1,0}.$$

Using the above expression for $\hat{\Delta}(\hat{a})$, the r.h.s. gives

$$\begin{aligned}
(\mu_0 \otimes \mu_0)((b \otimes c)(\mathcal{F}_1^{-1} \otimes \mathcal{F}_1^{-1})(\hat{\Delta}(\hat{a}))) &= \sum_i \mu_0(b \lambda_i) \mu_0(c \mathcal{F}_1^{-1}(w_0^* \gamma(\hat{a}) \lambda_i^* w_1)) \\
&= \mu_0(c \mathcal{F}_1^{-1}(w_0^* \gamma(\hat{a}) b w_1)),
\end{aligned}$$

where we used that λ_i is a Pimsner-Popa basis for $\mu_0 : \mathcal{A}_{-1,0} \rightarrow \mathbf{C}$. Now

$$\begin{aligned}
w_0^* \gamma(\hat{a}) b w_1 &= w_0^* \gamma(\hat{a}) w_{-3}^* w_{-2} b w_1 \\
&= w_{-1}^* w_0^* \gamma(\hat{a}) w_{-2} b w_1 \\
&= \mathcal{F}_1(\mathcal{F}_1^{-1}(\hat{a}) b).
\end{aligned}$$

Hence, using the trace property of μ_0 we get

$$(\mu_0 \otimes \mu_0)((b \otimes c)(\mathcal{F}_1^{-1} \otimes \mathcal{F}_1^{-1})(\hat{\Delta}(\hat{a}))) = \mu_0(bc \mathcal{F}_1^{-1}(\hat{a})),$$

i.e. the defining equation for $\hat{\Delta}(\hat{a})$. This concludes the proof of v) and hence of the Theorem 26. \square

The statements iii) and iv) of Theorem 26 imply

$$\mathcal{N} = \mathcal{A}'_{0,1} \cap \mathcal{M}_1$$

and since clearly

$$\mathcal{M}_1 = \mathcal{M} \vee \mathcal{A}_{0,1}$$

we have the

Corollary 27 $\mathcal{A}_{0,1}$ acts outerly on \mathcal{M} and $\mathcal{M}_1 = \mathcal{M} \triangleleft \mathcal{A}_{0,1}$ is a crossed product.

6 The Multiplicative Unitary W

In this section we finish our analysis showing that the coproduct Δ is actually a $*$ -algebra homomorphism $\Delta : \mathcal{A}_{-1,0} \rightarrow \mathcal{A}_{-1,0} \otimes \mathcal{A}_{-1,0}$ such that $(\mathcal{A}_{-1,0}, \Delta, S, \mu_0)$ becomes a discrete Kac algebra with antipode S and Haar state μ_0 . We also verify $\mathcal{A}_{-2,-1}$ to be the dual compact Kac algebra and rederive the pairing formula. Our methods will heavily rely on the results of S. Baaj and G. Skandalis, [Ba-Sk], who used the notion of a multiplicative unitary W as a basic tool. Given a C^* -Hopf algebra \mathcal{A} with coproduct Δ and Haar state μ_0 the operator W is defined on $L^2(\mathcal{A}, \mu_0) \otimes L^2(\mathcal{A}, \mu_0)$ by

$$W|a \otimes b\rangle := \Delta(a)|\mathbf{1} \otimes b\rangle, \quad a, b \in \mathcal{A}.$$

Then W is unitary and satisfies the pentagon identity

$$W_{23}W_{12} = W_{12}W_{13}W_{23}. \quad (26)$$

Conversely, given a unitary

$$W : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$$

on some Hilbert space \mathcal{H} , such that W satisfies the above pentagon identity (26) together with some regularity conditions, S. Baaj and G. Skandalis, [Ba-Sk], have shown how to recover in $\mathcal{L}(\mathcal{H})$ an underlying Hopf algebra \mathcal{A} and its dual $\hat{\mathcal{A}}$, such that W is given by the above formula.

This is the route we will follow. We first use our definition of Δ , equation (12), to define a candidate for W in terms of its matrix elements

$$\langle \lambda_k \otimes \lambda_l | W(\lambda_i \otimes \lambda_j) \rangle := (\mu_0 \otimes \mu_0)((\lambda_k^* \otimes \lambda_l^*)\Delta(\lambda_i)(\mathbf{1} \otimes \lambda_j)).$$

Now we use the trace property of μ_0 to get

$$= (\mu_0 \otimes \mu_0)((\lambda_k^* \otimes \lambda_j \lambda_l^*)\Delta(\lambda_i)),$$

and therefore by the definition of Δ

$$= \mu_0((\lambda_k^* * (\lambda_j \lambda_l^*))\lambda_i).$$

We will then show that W is indeed a multiplicative unitary and that the reconstruction procedure of [Ba-Sk] precisely reproduces $\mathcal{A} \equiv \mathcal{A}_{-1,0}$ and $\hat{\mathcal{A}} \equiv \mathcal{A}_{-2,-1}$ together with the

pairing leading to our coproduct Δ . We also verify that \hat{S} given in Definition 18 is indeed the antipode.

To simplify computations we first consider $\hat{W} := \text{Ad } J_{\mathcal{A}_{-1,0}}(W)$ which gives

$$\begin{aligned} \langle \lambda_k \otimes \lambda_l | \hat{W}(\lambda_i \otimes \lambda_j) \rangle &:= (\mu_0 \otimes \mu_0)((\lambda_k^* \otimes \lambda_l^*)(\mathbf{1} \otimes \lambda_j)\Delta(\lambda_i)) \\ &= \mu_0((\lambda_k^* * (\lambda_l^* \lambda_j))\lambda_i) \end{aligned} \quad (27)$$

where again we used the trace property of μ_0 , i.e. the fact that λ_i^* is also a Pimsner-Popa basis. Using Corollary 3, a simple computation shows that \hat{W} is given on $\mathcal{H} \otimes \mathcal{H}$ by

$$\hat{W}|\lambda_i \otimes \lambda_j \rangle = \sum_{k,l} z_{ij}^{kl} |\lambda_k \otimes \lambda_l \rangle \quad (28)$$

where $z_{ij}^{kl} \in \mathcal{M}_1 \cap \mathcal{M}' = \mathbf{C}$ is,

$$z_{ij}^{kl} = w_1^* w_1^* w_1^* \lambda_k^* \gamma(\lambda_l^* \lambda_j) w_{-1} \lambda_i w_1 w_1. \quad (29)$$

The above sum converges and we actually have

Theorem 28

$$\begin{aligned} \hat{W} : \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H} \\ |\lambda_i \rangle \otimes |\lambda_j \rangle &\mapsto \sum_{k,l} (w_1^* w_1^* w_1^* \lambda_k^* \gamma(\lambda_l^* \lambda_j) w_{-1} \lambda_i w_1 w_1) |\lambda_k \rangle \otimes |\lambda_l \rangle \end{aligned}$$

defines a multiplicative unitary matrix.

Proof :i) Unitarity

$$\begin{aligned} \|\hat{W}(a \otimes b)\|^2 &= \sum_{k,l} (w_1^* w_1^* w_1^* \lambda_k^* \gamma(\lambda_l^* b) w_{-1} a w_1 w_1)^* (w_1^* w_1^* w_1^* \lambda_k^* \gamma(\lambda_l^* b) w_{-1} a w_1 w_1) \\ &= \sum_{k,l} (w_1^* w_1^* a^* w_{-1}^* \gamma(b^* \lambda_l) \lambda_k w_1 w_1 w_1) (w_1^* w_1^* w_1^* \lambda_k^* \gamma(\lambda_l^* b) w_{-1} a w_1 w_1) \\ &= \sum_{k,l} (w_1^* w_1^* a^* w_{-1}^* \gamma(b^*) \lambda_k (\gamma(\lambda_l) w_1 w_1 w_1) (w_1^* w_1^* w_1^* \gamma(\lambda_l^*)) \lambda_k^* \gamma(b) w_{-1} a w_1 w_1) \end{aligned}$$

Using the intertwiner property of w_1 , i.e.

$$(\gamma(\lambda_l) w_1 w_1 w_1) = w_1 \lambda_l w_1 w_1$$

and Corollary 4, i.e.

$$w_1^* w_1^* a^* w_{-1}^* \gamma(b^*) \lambda_k w_1 \lambda_l w_1 = c w_1^*, \quad c \in \mathbf{C},$$

some of the factors w_1^*, w_1 cancel due to $w_1^* w_1 = \mathbf{1}$, see Corollary 1, and we get

$$\|\hat{W}(a \otimes b)\|^2 = \sum_{k,l} w_1^* w_1^* a^* w_{-1}^* \gamma(b^*) \lambda_k w_1 \lambda_l w_1 w_1^* \lambda_l^* w_1^* \lambda_k^* \gamma(b) w_{-1} a w_1 w_1.$$

λ_i being a Pimsner-Popa basis implies

$$\begin{aligned} &= w_1^* w_1^* a^* w_{-1}^* \gamma(b^*) \gamma(b) w_{-1} a w_1 w_1 \\ &= w_{-1}^* \gamma(b^*) \gamma(b) w_{-1} w_1^* a^* a w_1 \\ &= \mu_0(b^* b) \mu_0(a^* a) = \|a \otimes b\|^2 \end{aligned}$$

by Corollary 3. Similarly one proves

$$\|\hat{W}^*(a \otimes b)\|^2 = \mu_0(b^* b) \mu_0(a^* a).$$

ii) Pentagon Relation Using (28), the Pentagon relation (26) reads as follows,

$$z_{j_2, i_3}^{n_2, n_3} z_{i_1, i_2}^{n_1, j_2} = z_{j_1, j_2}^{n_1, n_2} z_{i_1, j_3}^{j_1, n_3} z_{i_2, i_3}^{j_2, j_3} \quad (30)$$

where the summation is taken over doubled indices. We compute the l.h.s. of (30)

$$\begin{aligned} & z_{j_2, i_3}^{n_2, n_3} z_{i_1, i_2}^{n_1, j_2} \\ &= \sum_{j_2} w_1^* w_1^* w_1^* \lambda_{n_2}^* \gamma(\lambda_{n_3}^* \lambda_{i_3}) w_{-1} \lambda_{j_2} w_1 w_1 w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{j_2}^* \lambda_{i_2}) w_{-1} \lambda_{i_1} w_1 w_1 \\ &= \sum_{j_2} w_1^* w_1^* w_1^* \lambda_{n_2}^* \gamma(\lambda_{n_3}^* \lambda_{i_3}) w_{-1} \lambda_{j_2} w_1 w_1 w_1^* w_1^* w_1^* \gamma(\lambda_{j_2}^*) \gamma(\lambda_{i_2}) \lambda_{n_1}^* w_{-1} \lambda_{i_1} w_1 w_1. \end{aligned}$$

Using the intertwiner property of w_1 , $w_1^* \gamma(\lambda_{j_2}^*) = \lambda_{j_2}^* w_1^*$ and applying Corollary 4 as in the above proof of unitarity, we get similarly

$$= w_1^* w_1^* w_1^* \lambda_{n_2}^* \gamma(\lambda_{n_3}^* \lambda_{i_3}) w_{-1} w_1^* \gamma(\lambda_{i_2}) \lambda_{n_1}^* w_{-1} \lambda_{i_1} w_1 w_1.$$

We shift w_1^* from the middle to the left using the intertwining property which leads to

$$= w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{n_2}^*) \gamma^2(\lambda_{n_3}^*) \gamma^2(\lambda_{i_3}) w_{-3} \gamma(\lambda_{i_2}) w_{-1} \lambda_{i_1} w_1 w_1,$$

where we made use of $[\gamma(a), \lambda_{n_1}^*] = 0$ for all $a \in \mathcal{M}$.

For the r.h.s. of (30) we first compute

$$\begin{aligned} & z_{j_1, j_2}^{n_1, n_2} z_{i_1, j_3}^{j_1, n_3} \\ &= \sum_{j_1} w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{n_2}^* \lambda_{j_2}) w_{-1} \lambda_{j_1} w_1 w_1 w_1^* w_1^* w_1^* \lambda_{j_1}^* \gamma(\lambda_{n_3}^* \lambda_{j_3}) w_{-1} \lambda_{i_1} w_1 w_1. \end{aligned}$$

Using similar arguments as before we get

$$= w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{n_2}^* \lambda_{j_2}) w_{-1} w_{-3} \gamma(\lambda_{n_3}^* \lambda_{j_3}) w_{-1} \lambda_{i_1} w_1 w_1.$$

Now we push w_{-3}^* to the left. Using $w_{-1} w_{-3}^* = w_{-5}^* w_{-1}$ and $(w_1^*)^3 w_{-5}^* = (w_1^*)^4$ we get

$$= w_1^* w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{n_2}^*) \gamma^2(\lambda_{n_3}^*) \gamma(\lambda_{j_2}) w_{-1} \gamma(\lambda_{j_3}) w_{-1} \lambda_{i_1} w_1 w_1. \quad (31)$$

Finally we compute $z_{i_2, i_3}^{j_2, j_3} = w_1^* z_{i_2, i_3}^{j_2, j_3} w_1$ where

$$\begin{aligned}
z_{i_2, i_3}^{j_2, j_3} &= w_1^* w_1^* \lambda_{j_2}^* \gamma(\lambda_{j_3}^* \lambda_{i_3}) w_{-1} \lambda_{i_2} w_1 \\
&= w_1^* w_1^* \gamma(\lambda_{j_3}^*) \lambda_{j_2}^* \gamma(\lambda_{i_3}) w_{-1} \lambda_{i_2} w_1 \\
&= w_1^* \lambda_{j_3}^* w_1^* \lambda_{j_2}^* \gamma(\lambda_{i_3}) w_{-1} \lambda_{i_2} w_1 \in \mathbf{C}
\end{aligned} \tag{32}$$

where we used the intertwiner property of w_1 . Now $z_{i_2, i_3}^{j_2, j_3} \in \mathcal{A}_{1,1} = \mathbf{C}$ and hence $z_{i_2, i_3}^{j_2, j_3} = \gamma(z_{i_2, i_3}^{j_2, j_3})$ and we conclude

$$z_{i_2, i_3}^{j_2, j_3} = w_{-1}^* \gamma(\lambda_{j_3}^*) w_{-1}^* \gamma(\lambda_{j_2}^*) \gamma^2(\lambda_{i_3}) w_{-3} \gamma(\lambda_{i_2}) w_{-1}.$$

Plugging this expression into the formula (32) between $\gamma(\lambda_{j_3}) w_{-1}$ and λ_{i_1} and using twice the property of λ_i being a Pimsner-Popa-basis we conclude

$$\begin{aligned}
z_{j_1, j_2}^{n_1, n_2} z_{i_1, j_3}^{j_1, n_3} z_{i_2, i_3}^{j_2, j_3} &= w_1^* w_1^* w_1^* \lambda_{n_1}^* \gamma(\lambda_{n_2}^*) \gamma^2(\lambda_{n_3}^*) \gamma^2(\lambda_{i_3}) w_{-3} \gamma(\lambda_{i_2}) w_{-1} \lambda_{i_1} w_1 w_1 \\
&= z_{j_2, i_3}^{n_2, n_3} z_{i_1, i_2}^{n_1, j_2}
\end{aligned}$$

which proves the Pentagon identity. \square

Let us remark here that the definition of the multiplicative matrix \hat{W} and its properties do not rely on the tracial property of μ_0 .

We get as a simple

Corollary 29 $W := \text{Ad } J_{\mathcal{A}_{-1,0}}(\hat{W})$ is a multiplicative unitary.

We now use the reconstruction theorems of S. Baaj and G. Skandalis, [Ba-Sk], to show that $\mathcal{A}_{-2,-1}$ and $\mathcal{A}_{-1,0}$ are indeed the pair of dual Hopf algebras associated with W . To this end we first perform the reconstruction starting from \hat{W} and then use $W = \text{Ad } J_{\mathcal{A}_{-1,0}}(W)$ to arrive at our final results.

For $a, b \in \mathcal{A}_{-1,0}$, let

$$\omega_{a,b} := \langle a | \cdot | b \rangle, \quad \in (\mathcal{A}_{-1,0})_*.$$

Following [Ba-Sk] we define

$$\begin{aligned}
\hat{\rho}(\omega_{a,b}) &:= (\mathbf{1} \otimes \omega_{a,b})(\hat{W}) \\
\hat{L}(\omega_{a,b}) &:= (\omega_{a,b} \otimes \mathbf{1})(\hat{W}).
\end{aligned}$$

The statement of S. Baaj and G. Skandalis, [Ba-Sk], is that the $*$ -closure of

$$\{\hat{\rho}(\omega_{a,b}) \in \mathcal{L}(\mathcal{H}) | a, b \in \mathcal{A}_{-1,0}\}$$

generate a Hopf-algebra with dual generated by

$$\{\hat{L}(\omega_{a,b}) \in \mathcal{L}(\mathcal{H}) | a, b \in \mathcal{A}_{-1,0}\}.$$

We compute

Lemma 30 For $a, b \in \mathcal{A}_{-1,0}$ we have

$$i) \quad \hat{\rho}(\omega_{a,b}) = \mathcal{F}(a^*b), \quad a, b \in \mathcal{A}_{-1,0},$$

$$ii) \quad \hat{L}(\omega_{a,b}) = \text{Ad } J_{\mathcal{A}_{-1,0}} \left(\sum_i \lambda_i w_{-1}^* \gamma(\lambda_i)^* w_{-1}^* \gamma(a)^* w_{-3} \gamma(b) w_{-1} \right)^* \in \mathcal{A}'_{-1,0} \cap \mathcal{L}(\mathcal{H})$$

for $a, b \in \mathcal{A}_{-1,0}$.

Proof : i) Let $a, b \in \mathcal{A}_{-1,0}$. We get from the definition

$$\begin{aligned} \hat{\rho}(\omega_{a,b}) &= \sum_{i,k} w_{-1}^* w_{-1}^* \lambda_k^* \gamma(a^*b) w_{-1} \lambda_i w_{-1} |\lambda_k \rangle \langle \lambda_i| \\ &= \sum_{i,k} w_{-1}^* \lambda_k^* (w_{-3}^* \gamma(a^*b) w_{-1}) \lambda_i w_{-1} |\lambda_k \rangle \langle \lambda_i| \\ &= w_{-3}^* \gamma(a^*b) w_{-1} = \mathcal{F}(a^*b) \in \mathcal{L}(\mathcal{H}). \end{aligned}$$

ii) A similar computation as in i) shows

$$\begin{aligned} \hat{L}(\omega_{a,b}) &= \sum_{j,l} w_1^* w_1^* \gamma(\lambda_l^* \lambda_j) a^* w_{-1} b w_1 |\lambda_l \rangle \langle \lambda_j| \\ &= \sum_{j,l} w_1^* \lambda_l^* \lambda_j w_1^* a^* w_{-1} b w_1 |\lambda_l \rangle \langle \lambda_j|. \end{aligned}$$

Now we use equality (16). Then we can rewrite

$$\begin{aligned} &= \sum_{j,l} w_1^* \lambda_l^* \lambda_j w_0^* w_1 w_1^* a^* w_{-1} b w_1 |\lambda_l \rangle \langle \lambda_j| \\ &= \sum_{j,l} w_1^* \lambda_l^* \lambda_j (w_0^* w_{-1}^* \gamma(a^*) w_{-3} \gamma(b) w_{-1}) w_1 |\lambda_l \rangle \langle \lambda_j| \end{aligned}$$

One easily shows

$$(w_0^* w_{-1}^* \gamma(a^*) w_{-3} \gamma(b) w_{-1}) \in \mathcal{M} \cap \gamma(\mathcal{M})' = \mathcal{A}_{-1,0},$$

see for example the computation in (10), and we get

$$\begin{aligned} \hat{L}(\omega_{a,b}) &= \sum_i |\lambda_i (w_0^* w_{-1}^* \gamma(a^*) w_{-3} \gamma(b) w_{-1}) \rangle \langle \lambda_i| \\ &= \text{Ad } J_{\mathcal{A}_{-1,0}} (w_0^* w_{-1}^* \gamma(a^*) w_{-3} \gamma(b) w_{-1})^*, \end{aligned}$$

which proves ii) . □

In order to recover our setting we now have to take the multiplicative unitary W instead of \hat{W} . This gives

$$\begin{aligned} \rho(\omega_{a,b}) &= J_{\mathcal{A}_{-1,0}} \hat{\rho}(\omega_{a^*,b^*}) J_{\mathcal{A}_{-1,0}} \\ L(\omega_{a,b}) &= J_{\mathcal{A}_{-1,0}} \hat{L}(\omega_{a^*,b^*}) J_{\mathcal{A}_{-1,0}} \end{aligned}$$

from which we conclude

$$\rho(\omega_{a,b}) = \mathcal{F}(ba^*) \in \mathcal{A}_{-2,-1} \quad (33)$$

$$L(\omega_{a,b}) = \sum_i w_{-1}^* \gamma(b) w_{-3}^* \gamma(a)^* w_{-1} \gamma(\lambda_i) w_{-1} \lambda_i^* \in \mathcal{A}_{-1,0}. \quad (34)$$

It is clear that the C^* -closures generated by these operators give $\mathcal{A}_{-2,-1}$ and $\mathcal{A}_{-1,0}$, respectively.

Next, according to [Ba-Sk] the natural pairing between the two algebras is given by

$$\langle L(\omega_{a,b}) | \rho(\omega_{c,d}) \rangle = \langle a \otimes c | W | b \otimes d \rangle = \mu_0(c^* L(\omega_{a,b}) d)$$

which by the trace property gives

$$\begin{aligned} &= \mu_0(dc^* L(\omega_{a,b})) \\ &= \mu_0(\mathcal{F}^{-1}(\rho(\omega_{c,d})) L(\omega_{a,b})) \end{aligned} \quad (35)$$

i.e. the natural pairing of our setting.

Finally we look at the antipode on $\mathcal{A}_{-2,-1}$ as defined by [Ba-Sk],

$$S_{B-S}(\rho(\omega_{c,d})) := (\mathbf{1} \otimes \omega_{c,d})(W^*).$$

Computing the r.h.s. similarly as before we get

$$S_{B-S}(\mathcal{F}(ba^*)) = \mathcal{F}(ab^*)^*,$$

which due to Definition 18 and equation (22) coincides with our antipode \hat{S} .

We conclude by mentioning without proof that we could have chosen to work with the opposite coproduct Δ_{op} on $\mathcal{A}_{-1,0}$. This would give W_{op} and \hat{W}_{op} , respectively. Putting $R = \hat{W}_{op}\sigma$, where σ denotes the flip on $\mathcal{H} \otimes \mathcal{H}$, we recover Longo's definition, [Lo1], of the R-operator of J. Cuntz, [Cu].

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